Optimization: from Static to Dynamic

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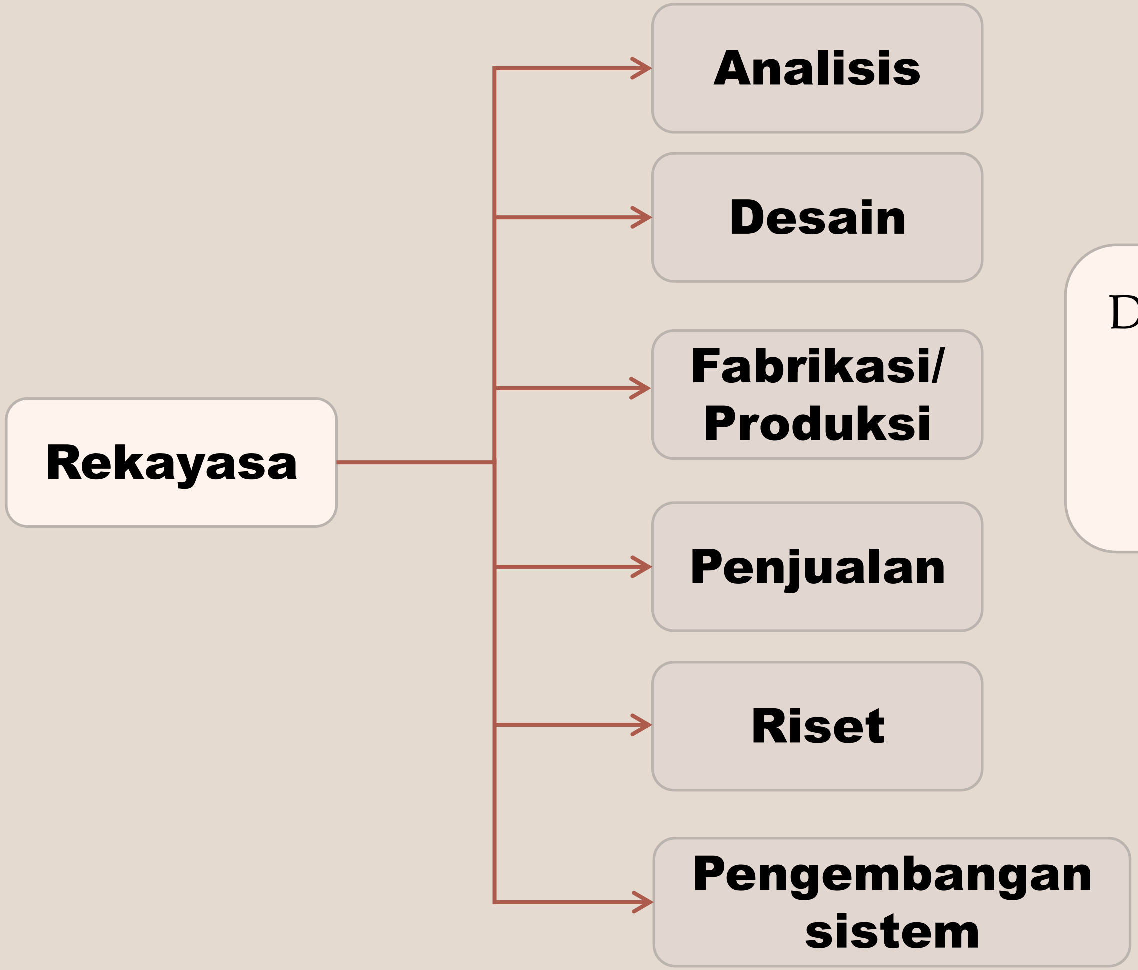
agenda

Optimasi ?



Desain sistem dapat diformulasikan sebagai permasalahan optimisasi dimana fungsi performa yang dioptimalkan harus memenuhi persyaratan-persyaratan tertentu

Perumusan Masalah

Langkah 1: Deskripsi proyek/masalah

Langkah 2: Pengumpulan data dan informasi

Langkah 3: Definisi variabel desain

Langkah 4: Perumusan fungsi tujuan

Langkah 5: Perumusan kendala

Bagian 1: Optimasi statis

Optimasi

- Pemrograman linier (Linear Programming)
- Pemrograman interger (Integer Programming)
- Pemrograman tak-linier (Nonlinear Programming)
- Pemodelan jaringan (Network Modelling)
- Teori persedian (Inventory Theory)
- Teori antrian (Queueing Theory)
- Teori permainan (Games Theory
- Teori keputusan (Decision Theory)
- Pemrograman dinamik (Dynamics Pemrograming)
- Proses Markov (Markov Process)

Contoh: Penentuan Lokasi Toko Alat Tulis

Seorang ingin mengembangkan bisnis toko alat tulis. Rencananya akan membuka toko alat tulis di sekitar sekolah dasar dan sekolah menengah. Di peta koordinat, keduanya sekolah terletak di titik A (20, 0) dan B (0, 20). Di titik (0, 0), ada toko dengan karakteristik yang sama dan, menurut undang-undang kota, posisi toko alat tulis baru harus berada pada jarak minimal 400 m dari yang sudah ada.

Tentukan titik di mana toko alat tulis baru harus berada, minimalkan jumlah jarak ke kuadrat dari dua sekolah.

Penyelesaian

Variabel keputusan:

X Koordinat pada sumbu-x dimana lokasi toko alat tulis berada

Y Koordinat pada sumbu-y dimana lokasi toko alat tulis berada

Penyelesaian: Penentuan Lokasi Toko Alat Tulis

Pemodelan matematika

Penyelesaian

Variabel keputusan:

x posisi pada sumbu-x dimana lokasi toko alat tulis berada

y posisi pada sumbu-y dimana lokasi toko alat tulis berada

Fungsi tujuan (Objective function)

$$\begin{aligned}\min f(x,y) &= (x-0)^2 + (y-20)^2 + (x-20)^2 + (y-0)^2 \\ &= 2(x^2+y^2) - 40(x+y) + 800\end{aligned}$$

Kendala

$$x^2 + y^2 \geq 400^2 \rightarrow g:\ 400^2 - x^2 - y^2 \leq 0$$

Pemrograman Tak-linier

Fungsi tujuan

$$\min f(\boldsymbol{x})$$

dengan kendala

$$c(x) \le 0,\quad ceq(x) = 0,\quad A.x \le b,\quad Aeq.x = beq,\quad lb \le x \le ub$$

$$\min f(x) = 2(x^2 + y^2) - 40(x + y) + 800$$

$$g:\ 400^2 - x^2 - y^2 \le 0$$

Penyelesaian: Penentuan Lokasi Toko Alat Tulis

Permasalahan Pemrograman tak Linier dapat diselesaikan dengan menerapkan kondisi **Karush-Kuhn-Tucker**:

Langkah 1: membuktikan bahwa f dan g memenuhi partial continuous first-order derivatives.

Langkah 2: Bentuk fungsi Lagrangian:

$$L(x, y, \lambda) = (x-0)^2 + (y-20)^2 + (x-20)^2 + (y-0)^2 + \lambda(400^2 - x^2 - y^2)$$

Langkah 3: Bentuk sistem persamaan dan pertidaksamaan aljabar (Karush-Kuhn-Tucker)

$$\frac{\partial L}{\partial x} = 4x - 40 - 2\lambda x = 0$$

$$\frac{\partial L}{\partial y} = 4y - 40 - 2\lambda y = 0$$

$$\frac{\partial L}{\partial \lambda} = g : \ 400^2 - x^2 - y^2 \leq 0$$

$$\lambda(400^2 - x^2 - y^2) = 0$$

$$\lambda \geq 0$$

Jika $\lambda = 0$, maka $x = y = 10$, dan kendala tidak terpenuhi (dimana $400^2 - x^2 - y^2 > 0$)

Jika $\lambda \neq 0$, maka diperoleh $x = y = 282{,}84$ dan $\lambda = 1.93$

Optimization Toolbox: fmincon

```
>> x0=[100,100];
>> [x, fval] = fmincon(@example,x0,[],[],[],[],[],[],@conexp)

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>

x =

  282.8427   282.8427
```

```
function [c, ceq] = conexp(x)
c = [400^2-x(1)*x(1)-x(2)*x(2)];
ceq = [];
```

Kendala

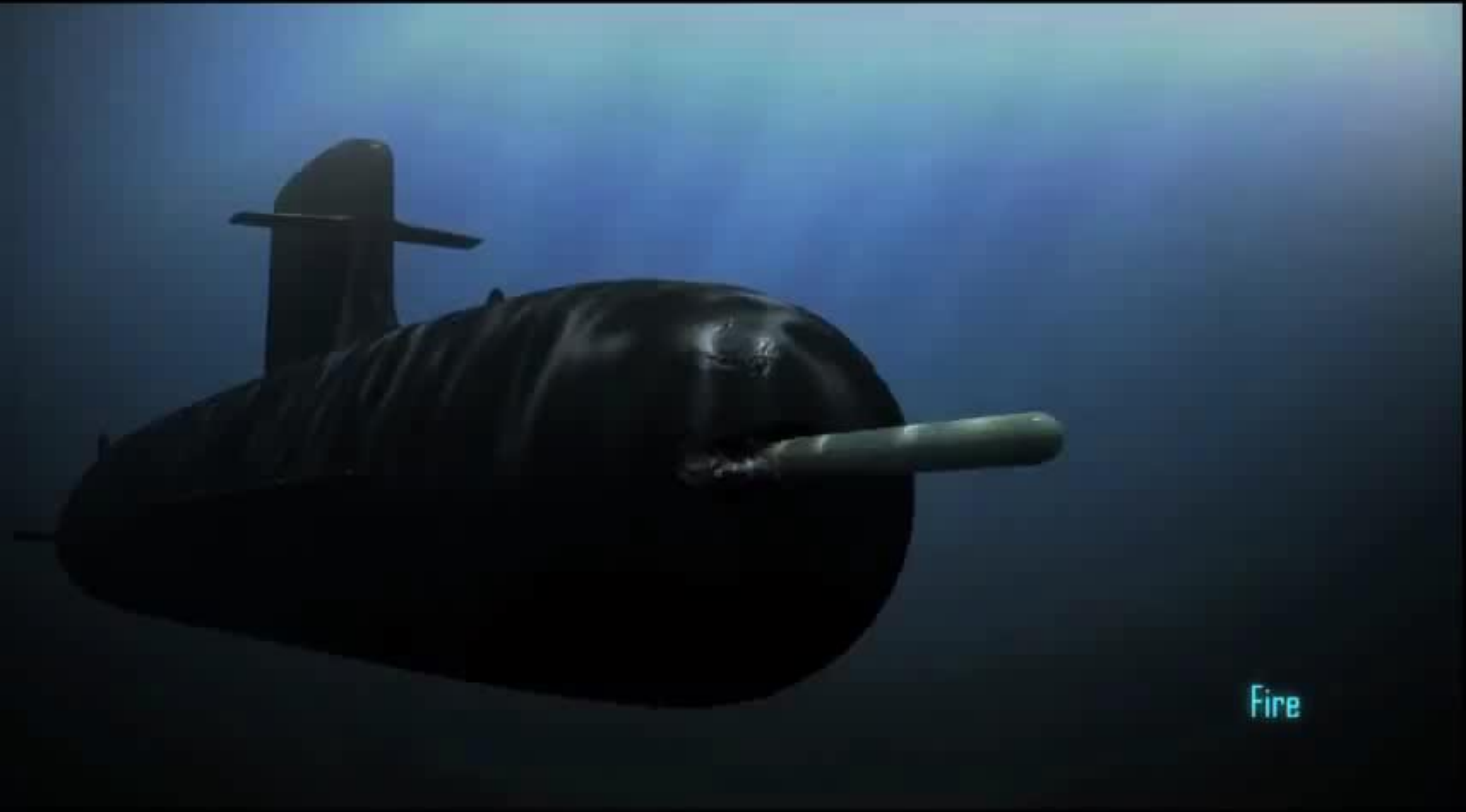
```
function f = example(x)
f = 2*(x(1)*x(1)+x(2)*x(2))-40*(x(1)+x(2))+800;
```

Fungsi tujuan

Matlab Optimization Toolbox

Problem type	Formulation	MATLAB function
One-variable minimization in fixed interval	Find $x \in [x_L \; x_U]$ to minimize $f(x)$	`fminbnd`
Unconstrained minimization	Find $\mathbf{x}$ to minimize $f(\mathbf{x})$	`fminunc` `fminsearch`
Constrained minimization: Minimize a function subject to linear inequalities and equalities, nonlinear inequalities and equalities, and bounds on the variables	Find x to minimize $f(\mathbf{x})$ subject to $\mathbf{Ax} \le \mathbf{b}$, $\mathbf{Nx} = \mathbf{e}$ $g_i(\mathbf{x}) \le 0$, $i = 1$ to m $h_j = 0$, $j = 1$ to p $x_{iL} \le x_i \le x_{iU}$	`fmincon`
Linear programming: minimize a linear function subject to linear inequalities and equalities	Find x to minimize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} \le \mathbf{b}$, $\mathbf{Nx} = \mathbf{e}$	`linprog`
Quadratic programming: Minimize a quadratic function subject to linear inequalities and equalities	Find $\mathbf{x}$ to minimize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{Hx}$ subject to $\mathbf{Ax} \le \mathbf{b}$, $\mathbf{Nx} = \mathbf{e}$	`quadprog`

Bagian 2: Optimasi dinamik


Fire

Guided Missile

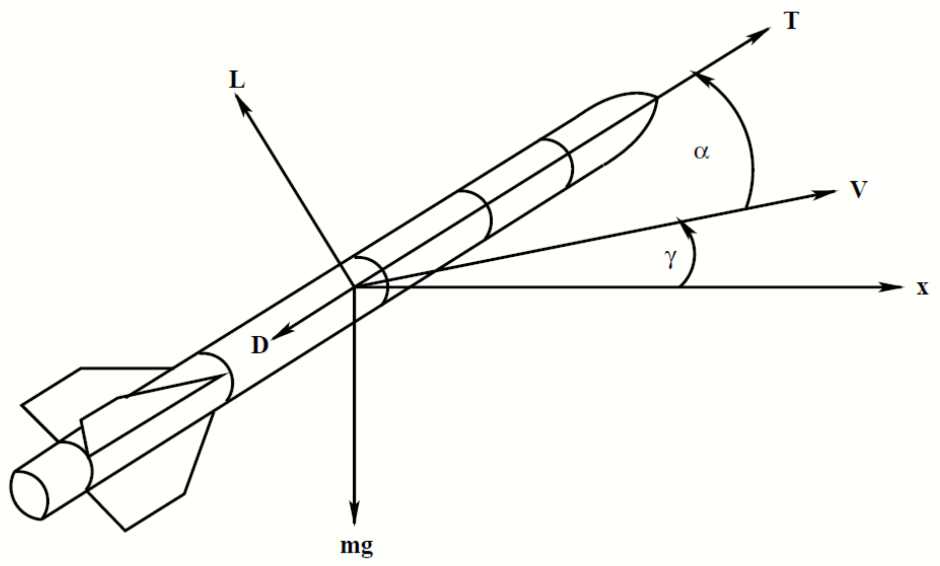
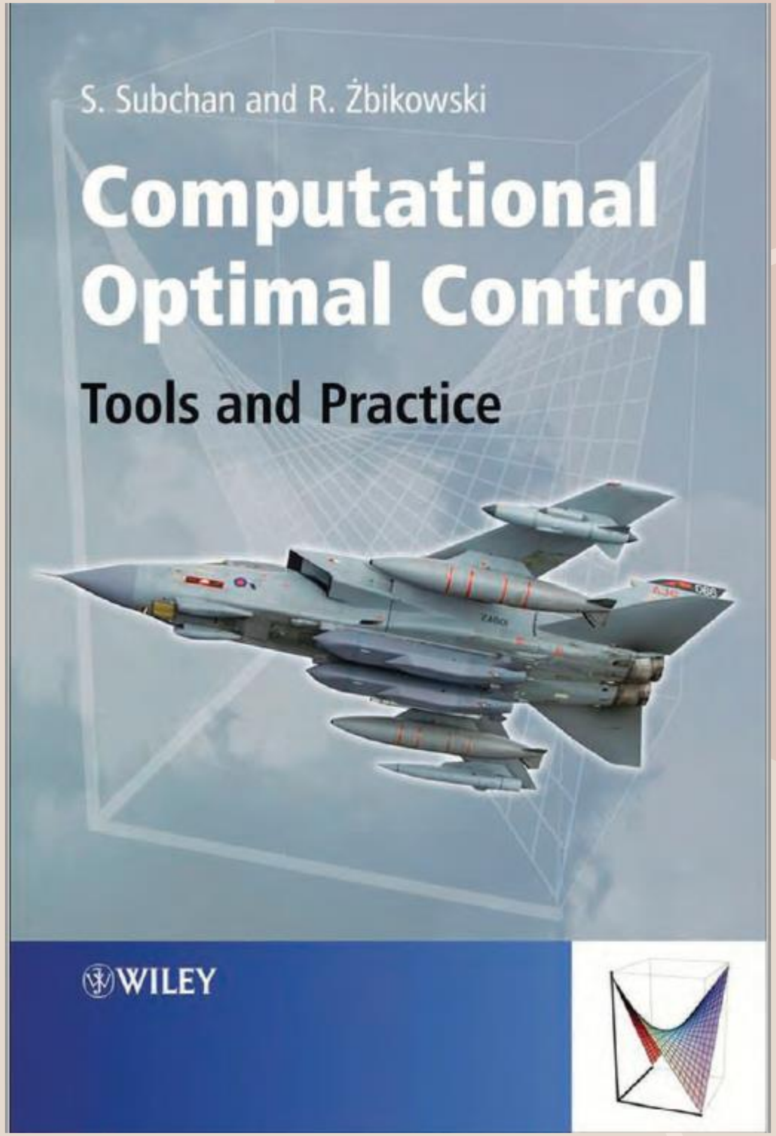


Figure 1.1 Definition of missile axes and angles. Note that L is the normal aerodynamic force and D is the axial aerodynamic force with respect to a body-axis frame, not lift and drag.



Pemodelan Matematika

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$$J = \int_{t_0}^{t_f} h \, dt,$$

Fungsi tujuan

$$J = \int_{t_0}^{t_f} dt.$$

$$\dot{\gamma} = \frac{T - D}{mV} \sin\alpha + \frac{L}{mV} \cos\alpha - \frac{g \cos\gamma}{V}$$

$$\dot{V} = \frac{T - D}{m} \cos\alpha - \frac{L}{m} \sin\alpha - g \sin\gamma$$

$$\dot{x} = V \cos\gamma$$

$$\dot{h} = V \sin\gamma$$

sistem dinamik

$$\gamma(0) = \gamma_0, \qquad \gamma(t_f) = \gamma_{t_f}$$

$$V(0) = V_0, \qquad V(t_f) = V_{t_f}$$

$$x(0) = x_0, \qquad x(t_f) = x_{t_f}$$

$$h(0) = h_0, \qquad h(t_f) = h_{t_f}.$$

kondisi batas

Masalah Nilai Batas (Two-point BVP)

Penyelesaian Optimasi Dinamik

Diberikan sistem dinamik

$$\dot{x} = f(x(t), u(t), t)$$

dengan fungsi tujuan

$$\min J = \int_0^T V(x(t), u(t), t)dt + S(x(T), T)$$

dan kondisi batas

$$x(0) = x_0, x(T) = x_T$$

Penyelesaian Optimasi Dinamik

Dari permasalahan yang telah diberikan sebelumnya, permasalahan tersebut dapat diselesaikan sebagaimana diberikan pada langkah-langkah berikut.

1. Mengonstruksi fungsi Pontryagin $\mathcal{H}$

$$\mathcal{H}(x(t), u(t), \lambda(t), t) = V(x(t), u(t), t) + \lambda'(t) f(x(t), u(t), t)$$

2. Meminimumkan $\mathcal{H}$ terhadap $u(t)$,

$$\left(\frac{\partial \mathcal{H}}{\partial u}\right)_* = 0 \quad \text{dan diperoleh} \quad u^*(t) = h(x^*(t), \lambda^*(t), t)$$

3. Menerapkan hasil pada tahap 2 untuk memperoleh $\mathcal{H}$ pada tahap 1,

$$\mathcal{H}^*(x^*(t), h(x^*(t), \lambda^*(t), t), \lambda^*(t), t) = \mathcal{H}^*(x^*(t), \lambda^*(t), t)$$

Penyelesaian Optimasi Dinamik

4. Menyelesaikan dua persamaan diferensial berikut

$$\dot{x}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda}\right)_*$$

$$\dot{\lambda}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x}\right)_*$$

dengan kondisi awal x_0 dan kondisi akhir

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \partial t_f + \left[\left(\frac{\partial S}{\partial x}\right)_* - \lambda^*(t)\right]_{t_f}' \partial x_f = 0$$

5. Substitusikan hasil dari $x^*(t), \lambda^*(t)$ pada tahap 4 ke persamaan kontrol optimal $u^*(t)$ pada tahap 2.

Tipe-tipe Sistem Dinamik

No.	Tipe	Substitusi	Kondisi Batas
1	*Fixed-final time and fixed-final state system*	$\partial t_f = 0, \partial x_f = 0$	$x(t_0) = x_0, \quad x(t_f) = x_f$
2	*Free-final time and fixed-final state system*	$\partial t_f \neq 0, \partial x_f = 0$	$x(t_0) = x_0,\ x(t_f) = x_f, \left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0$
3	*Fixed-final time and free-final state system*	$\partial t_f = 0, \partial x_f \neq 0$	$x(t_0) = x_0, \lambda^*(t_f) = \left(\frac{\partial S}{dx}\right)_{*t_f}$
4	*Free-final time and dependent free-final state system*	$\partial x_f = \dot{\theta}(t_f)\partial t_f$	$x(t_0) = x_0, \quad x(t_f) = \theta(t_f), \left[\mathcal{H}^* + \frac{\partial S}{\partial t} + \left\{\left(\frac{\partial S}{\partial x}\right)_* - \lambda^*(t)\right\}' \dot{\theta}(t)\right]_{t_f} = 0$
5	*Free-final time and independent free-final state system*	$\partial t_f = 0, \partial x_f \neq 0$	$\partial x(t_0) = x_0, \left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0, \left[\left(\frac{\partial S}{\partial x}\right)_* - \lambda^*(t_f)\right]_{t_f} = 0$

Contoh

Diberikan sistem dinamik

$$\dot{x}_1(t) = x_2(t); \dot{x}_2(t) = u(t)$$

dengan fungsi tujuan

$$\min J = \frac{1}{2}\int_0^2 u^2(t)dt$$

dan kondisi batas

$$x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Penyelesaian-1

Dari permasalahan yang telah diberikan sebelumnya, permasalahan tersebut dapat diselesaikan sebagaimana diberikan pada langkah-langkah berikut.

1. Mengonstruksi fungsi Pontryagin $\mathcal{H}$

$$\begin{aligned}\mathcal{H}(x(t), u(t), \lambda(t), t) &= V(x(t), u(t), t) + \lambda'(t) f(x(t), u(t), t) \\ &= \frac{1}{2}\mathrm{u}^2(t) + \lambda_1(t) x_2(t) + \lambda_2(t) u(t)\end{aligned}$$

2. Meminimumkan

$$\left(\frac{\partial \mathcal{H}}{\partial u}\right)_* = u(t) + \lambda_2(t) = 0$$

dan diperoleh

$$u^*(t) = -\lambda_2(t)$$

3. Menerapkan hasil pada tahap 2 untuk memperoleh $\mathcal{H}$ pada tahap 1,

$$\mathcal{H}^*(x^*(t), h(x^*(t), \lambda^*(t), t), \lambda^*(t), t) = \lambda_1(t) x_2(t) - \frac{1}{2}\lambda_2^2(t)$$

Penyelesaian-2

$$\dot{x}_1^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda_1}\right)_* = x_2^*(t)$$
$$\dot{x}_2^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda_2}\right)_* = -\lambda_2^*(t)$$
$$\dot{\lambda}_1^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_1}\right)_* = 0$$
$$\dot{\lambda}_2^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_2}\right)_* = -\lambda_1^*(t).$$

$$x_1^*(t) = \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1$$
$$x_2^*(t) = \frac{C_3}{2}t^2 - C_4 t + C_2$$
$$\lambda_1^*(t) = C_3$$
$$\lambda_2^*(t) = -C_3 t + C_4.$$

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

$$x(0) = \begin{pmatrix}1\\2\end{pmatrix}, x(2) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$x_1^*(t) = 0.5t^3 - 2t^2 + 2t + 1,$$
$$x_2^*(t) = 1.5t^2 - 4t + 2,$$
$$\lambda_1^*(t) = 3,$$
$$\lambda_2^*(t) = -3t + 4,$$
$$u^*(t) = 3t - 4.$$

Pemodelan Matematika

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Fungsi tujuan

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$$\dot{V} = \frac{T-D}{m}\cos\alpha - \frac{L}{m}\sin\alpha - g\sin\gamma$$

$$\dot{x} = V\cos\gamma$$

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sistem dinamik

$$\gamma(0) = \gamma_0, \qquad \gamma(t_f) = \gamma_{t_f}$$

$$V(0) = V_0, \qquad V(t_f) = V_{t_f}$$

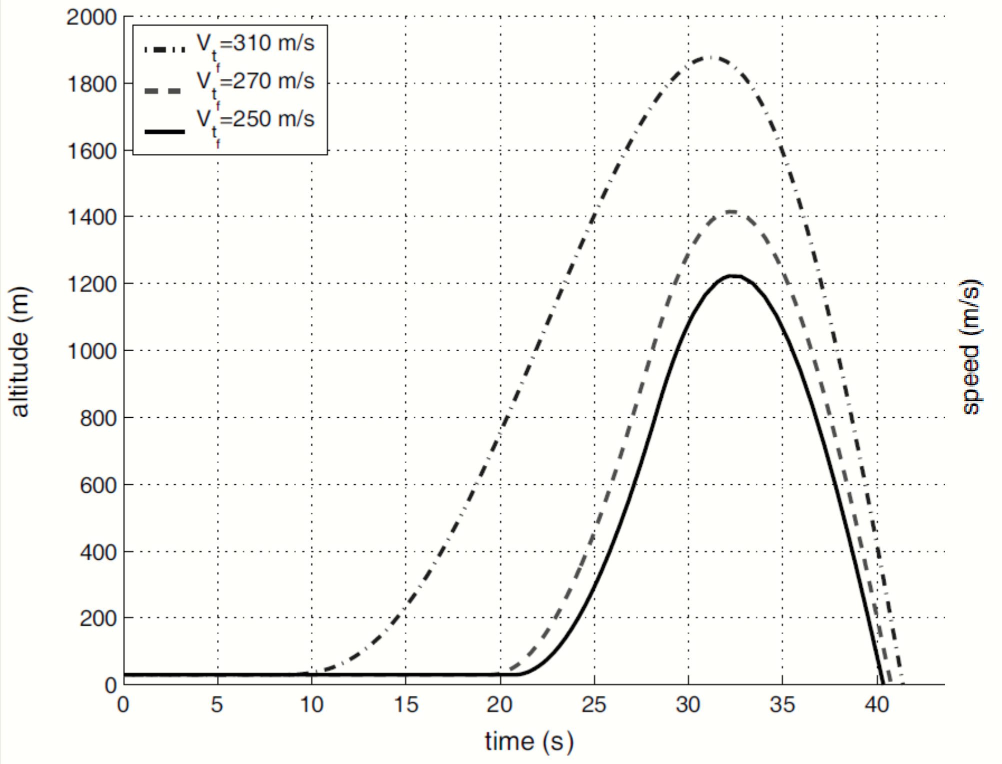
$$x(0) = x_0, \qquad x(t_f) = x_{t_f}$$

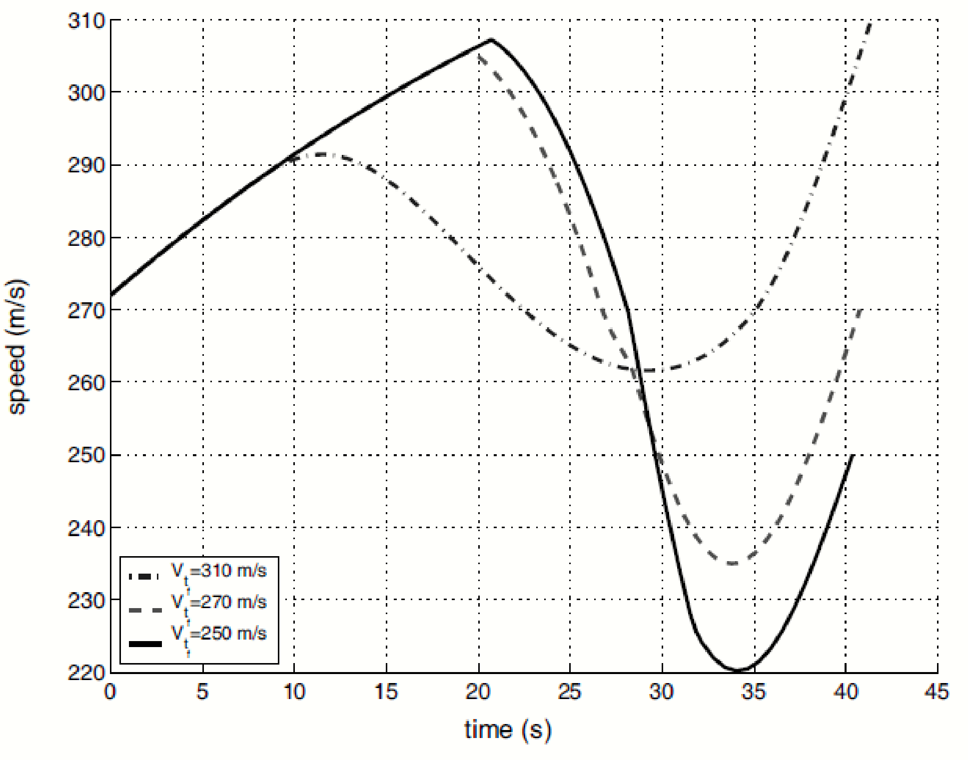
$$h(0) = h_0, \qquad h(t_f) = h_{t_f}.$$

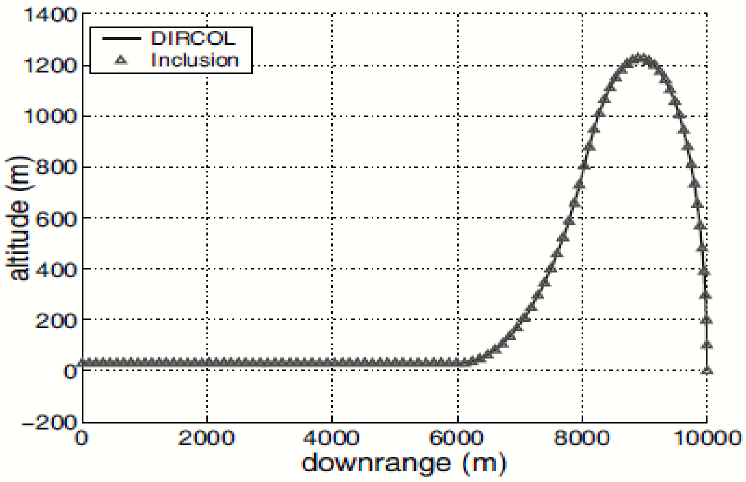
kondisi batas

Masalah Nilai Batas (Two-point BVP)

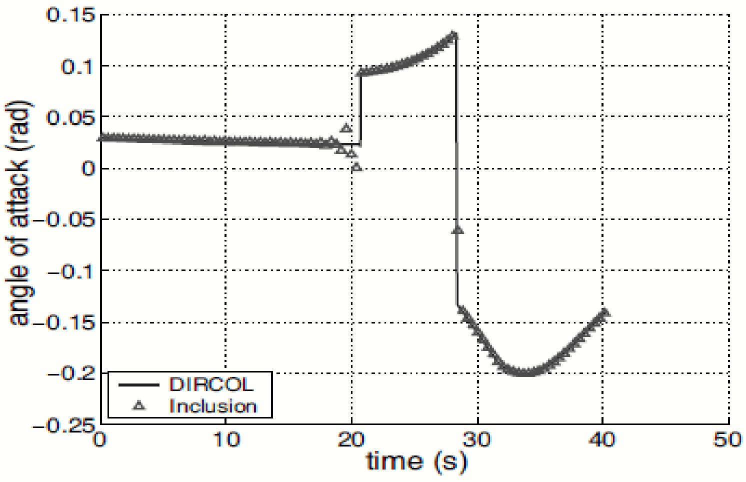
Hasil komputasi



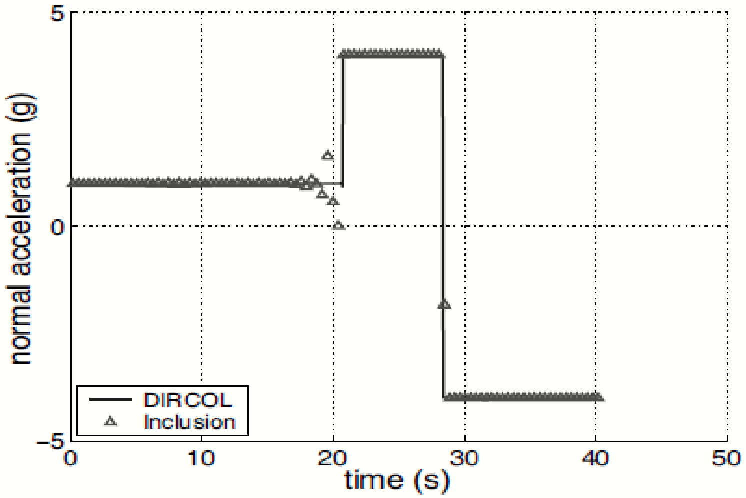




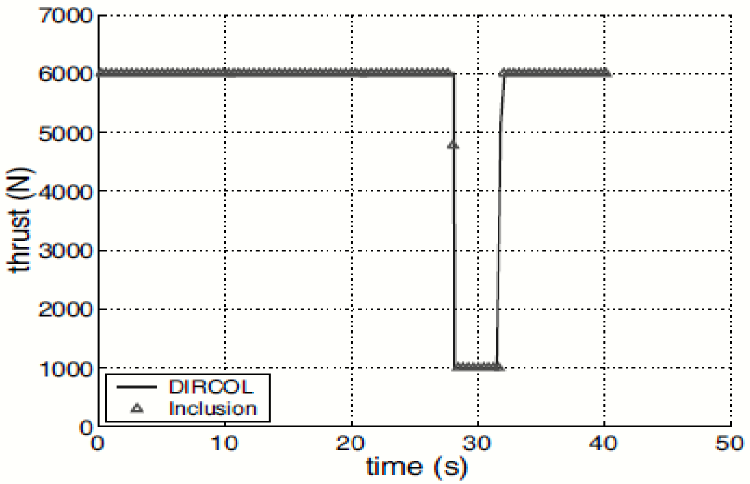
(a) Altitude versus downrange



(b) Angle of attack versus time



(c) Normal acceleration versus time



(d) Thrust versus time

thank you

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